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Quantum Stirling heat engine and refrigerator with single and coupled spin systems

Xiao-Li Huang[1,a], Xin-Ya Niu[1], Xiao-Ming Xiu[2,3], and Xue-Xi Yi[2,b]

[1] School of Physics and Electronic Technology, Liaoning Normal University, Dalian 116029, P.R. China
[2] School of Physics and Optoelectronic Technology, Dalian University of Technology, Dalian 116024, P.R. China
[3] College of Mathematics and Physics, Bohai University, Jinzhou 121013, P.R. China

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Abstract. We study the reversible quantum Stirling cycle with a single spin or two coupled spins as the working substance. With the single spin as the working substance, we find that under certain conditions the reversed cycle of a heat engine is NOT a refrigerator, this feature holds true for a Stirling heat engine with an ion trapped in a shallow potential as its working substance. The efficiency of quantum Stirling heat engine can be higher than the efficiency of the Carnot engine, but the performance coefficient of the quantum Stirling refrigerator is always lower than its classical counterpart. With two coupled spins as the working substance, we find that a heat engine can turn to a refrigerator due to the increasing of the coupling constant, this can be explained by the properties of the isothermal line in the magnetic field-entropy plane.

1 Introduction

In thermodynamics, a cycle consists of a collection of thermodynamic processes transferring heat and work, while changing pressure, temperature, and other state variables, eventually returning a system to its initial state. In the process of going through this cycle, the system may perform the conversion of heat or thermal energy to mechanical work, named as a heat engine. Heat engines distinguish themselves from other types of engines by the fact that their efficiency is fundamentally limited by Carnot's theorem. There are four basic thermodynamical processes in classical thermodynamics, i.e., isothermal process, isochoric process, adiabatic process and isobaric process, they correspond to fixed temperature, volume (the generalized coordinate of the system), entropy, and pressure (the generalized force conjugated to the generalized coordinate), respectively. For example, an isothermal process is a change of the system, in which the temperature remains constant. This typically occurs when the system is in contact with an thermal reservoir, and the change occurs slowly enough to allow the system to continually adjust to the temperature of the reservoir through heat exchange. In contrast, an adiabatic process is where the system exchanges no heat with its surroundings. The working substance can be any system with a non-zero heat capacity, then one may wonder: what happens with a quantum system as the working substance. These questions were answered in references [1–6], in which four classical processes have been generalized to the quantum case. With these generalizations, all the classical thermodynamical cycles, such as Carnot cycle, Otto cycle, Brayton cycle, Diesel cycle, and Stirling cycle [7], can be generalized to the quantum case. The working substance for a quantum heat engine or quantum refrigerator can be all kinds of quantum systems, ranging from single spin [8–11], two-level or multilevel systems [12–16], harmonic-oscillators [17] to coupled spins [18–20]. Based on these working substance, the properties for quantum Carnot cycle [21–23], quantum Otto cycle [19,24], quantum Brayton cycle [20,25], and quantum Diesel cycle [6] have be considered from different points of view including the positive work condition [3,4,15], efficiency and power [26–28]. Apart from the pure academic interest, the quantum generalization of those thermodynamical processes may find applications. In fact, several schemes have been proposed to realize the quantum heat engine, some of them are realized in experiments. For example, driven coupled spins linked to two split heat baths can serve as a quantum Carnot heat engine by adjusting the energy difference [29]. The quantum Carnot cycle may also be realized by a single spin coupled to an oscillator and sandwiched between two heat baths by designing the dissipative process [30]. Recently, experiment to realize quantum Otto cycle are presented by confining a single ion in a tapered linear Paul trap and by coupling the ion to engineered laser reservoirs [31], or by cold bosonic atoms confined to a double well potential [32]. The main motivations about this topic can be summarized as follows. First, with the development of quantum

[a] e-mail: ghost820521@163.com
[b] e-mail: yixx@dlut.edu.cn

information processing and nanotechnology, one may manipulate and control a system at the nanoscale. At the nanoscale, quantum effects dominate. This invokes people to consider the thermodynamics of small quantum systems. Second, the study of the quantum heat engine provides us a way to better the work extraction and a method to improve the efficiency of heat engine beyond the classical limit [33,34]. Quantum resources, such as quantum coherence and entanglement [35], are resources which have no classical counterpart. It has been reported that the use of these quantum resources can strikingly affect the thermodynamics. For example, we can extract work from a single heat bath by vanishing quantum coherence [36–38], we can obtain a heat engine with efficiency higher than the Carnot one [39] by virtue of squeezed heat bath. Last but may be the most important, these studies provide us with a new viewpoint to better our understanding of the fundamental concept, such as Maxwell's demon and the universality of the second law [40–42], and the third law of thermodynamics [43,44], in thermodynamics, statistical physics and quantum mechanics [45–52].

In the previous works, the focus is on the cycles where there are two quantum adiabatic processes. There exists cycles in classical thermodynamics that have no adiabatic process. Then a question arises: what will happen when this type of cycle is generalized to the quantum domain. In this Manuscript, we study the quantum Stirling heat engine and refrigerator, which is an extension of the Stirling cycle from classical thermodynamics to quantum physics. An ideal Stirling cycle consists of two isochoric processes and two isothermal processes. The Stirling cycle is reversible, which means that if supplied with mechanical power, it can function as a heat pump for heating or cooling. We will show that there are many interesting properties which are different from its classical counterpart. For example, with a single spin-$\frac{1}{2}$ as the working substance, we show that under certain conditions, the reverse cycle of a heat engine is not a refrigerator, because it absorbs heat from the bath at high temperature and releases heat to the bath at low temperature, while the work is done on the working substance. This feature can be observed in the heat engines with an ion trapped in a shallow potential as its working substance. In addition, we find that with two coupled spins as the working substance, a heat engine can be turned into a refrigerator by increasing the coupling constant.

2 Single spin

A quantum Stirling cycle is a quantum counterpart of classical Stirling cycle, which consists of two quantum isothermal processes and two quantum isochoric processes. In the quantum isothermal process, the working substance is kept in contact with a heat bath at a constant temperature T and the system remains in thermal equilibrium with the heat bath at every instant of time. The generalized coordinate of the working substance is changed slowly, as a result, the energy spectrum E_n and the occupation probabilities p_n, as well as the internal energy U are changed.

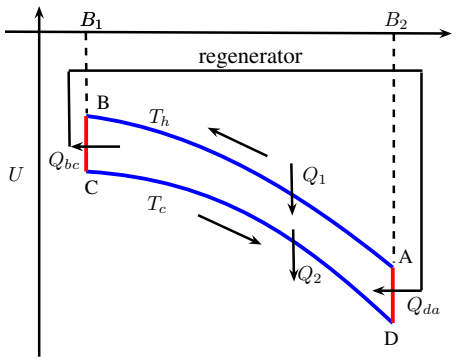

Fig. 1. Scheme illustration of a quantum Stirling heat engine cycle based on magnetic field-energy B-U diagram.

For a quantum system at thermal equilibrium, the occupation probabilities p_n are given by $p_n = \frac{1}{\mathcal{Z}}\exp(-\beta E_n)$, where $\mathcal{Z}$ is the partition function, H is the Hamiltonian of the system, $\beta = \frac{1}{kT}$ with k the Boltzmann constant. It should be noted that in the classical isothermal process, where the ideal gas is taken as the working substance, the internal energy does not change, but it is not true for quantum isothermal process. In classical isochoric process, the volume V, i.e., the generalized coordinate is fixed. Accordingly, in a quantum isochoric process, the generalized coordinate of the working substance is fixed. As a result, the energy spectrum of the system keeps unchanged during the quantum isochoric process and no work is done. For the detailed discussions about these generalizations, we refer the reader to reference [5].

The Hamiltonian of the working substance under consideration can be written as $H_s = \frac{1}{2}B\sigma^z$, where σ^z is the usual Pauli operator and B is the external magnetic field. There is only one parameter B in the Hamiltonian (i.e., the generalized coordinate), hence the quantum isochoric process for the working substance means that the magnetic field keeps as a constant and the magnetic field exerting on the working substance is a quantum analogy of the volume in the classical working substance (e.g., ideal gas). With these knowledge, the four stages for the heat engine can be depicted as follows. *Stage 1*: $A \to B$ is an isothermal process. In this process, the system interacts with an heat bath at high temperature T_h. The magnetic field changes from B_2 to B_1 slowly such that the system is at thermal equilibrium with the heat bath at any instance of time. Heat is absorbed from the bath (named Q_1). *Stage 2*: $B \to C$ is a quantum isochoric process. In this process, the magnetic field keeps B_1 and the temperature of the system decreases from T_h to T_c. No work is done and heat is released (named Q_{bc}) in this process. *Stage 3*: $C \to D$ is another isothermal process. The magnetic field changes from B_1 to B_2. The temperature of the system keeps T_c. Heat is released to the heat bath (named Q_2). *Stage 4*: $D \to A$ is another isochoric process. Heat is absorbed (named Q_{da}) and there is no work. A schematic diagram for this heat engine is given in Figure 1. All the above heat exchanges can be calculated as follows. In the two isothermal processes, we have $đQ = TdS$, where S

is the entropy of the working substance. For a two-level system at thermal equilibrium with temperature T, the entropy is:

$$S = -\sum_n p_n \ln p_n = k\left[\ln\left(2\cosh\frac{\beta B}{2}\right) - \frac{\beta B}{2}\tanh\frac{\beta B}{2}\right]. \tag{1}$$

Hence Q_1 and Q_2 can be obtained

$$\begin{aligned} Q_1 &= \int_A^B TdS = T_h(S(B)-S(A)) \\ &= kT_h \ln\frac{\cosh\frac{\beta_h B_1}{2}}{\cosh\frac{\beta_h B_2}{2}} + \frac{B_2}{2}\tanh\frac{\beta_h B_2}{2} - \frac{B_1}{2}\tanh\frac{\beta_h B_1}{2}, \end{aligned} \tag{2}$$

$$\begin{aligned} Q_2 &= \int_C^D TdS = T_c(S(D)-S(C)) \\ &= kT_c \ln\frac{\cosh\frac{\beta_c B_2}{2}}{\cosh\frac{\beta_c B_1}{2}} + \frac{B_1}{2}\tanh\frac{\beta_c B_1}{2} - \frac{B_2}{2}\tanh\frac{\beta_c B_2}{2}. \end{aligned} \tag{3}$$

In the two quantum isochoric processes, no work is done. The heat transfer equals to the change of the internal energy U. For our system, the internal energy reads:

$$U = \langle H\rangle = \sum_n p_n E_n = -\frac{B}{2}\tanh\frac{\beta B}{2}. \tag{4}$$

As a result, Q_{bc} and Q_{da} are:

$$Q_{bc} = U(C) - U(B) = \frac{B_1}{2}\left(\tanh\frac{\beta_h B_1}{2} - \tanh\frac{\beta_c B_1}{2}\right), \tag{5}$$

$$Q_{da} = U(A) - U(D) = \frac{B_2}{2}\left(\tanh\frac{\beta_c B_2}{2} - \tanh\frac{\beta_h B_2}{2}\right), \tag{6}$$

where $Q > 0$ and $Q < 0$ correspond to absorption and release of heat, respectively. Based on these heat exchanges, we can obtain the work output per cycle according to the first law of thermodynamics as:

$$W = \sum Q = kT_h \ln\frac{\cosh\frac{\beta_h B_1}{2}}{\cosh\frac{\beta_h B_2}{2}} + kT_c \ln\frac{\cosh\frac{\beta_c B_2}{2}}{\cosh\frac{\beta_c B_1}{2}}. \tag{7}$$

In classical Stirling heat engine, a regenerator is used in order to improve the performance of the cycle in the two isochoric processes. The regenerator is applied to the quantum Stirling heat engine too. This means that the heat released in Stage 2 is sent back in Stage 4. As a result, Stirling cycle is usually categorized as regenerative cycle. For classical ideal gas, we always have $|Q_{bc}| = |Q_{da}|$ (or $Q_{bc} + Q_{da} = 0$). However, it is not the case for the cycle with quantum working substance. Hence a net amount of heat transfer between the system and regenerator in the two isochoric processes is needed as:

$$\Delta Q = Q_{bc} + Q_{da}. \tag{8}$$

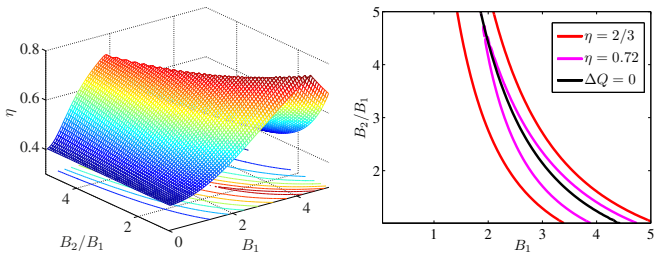

Fig. 2. Heat engine efficiency η as functions of B_1 (in units of kT_c) and B_2/B_1. Other parameters are $kT_h = 3kT_c$. Here $\eta = \frac{2}{3}$ is the classical Carnot efficiency under this condition.

Here we have three possible cases: (i) $\Delta Q = 0$. This is the condition for perfect regeneration. As discussed above, this is true for classical ideal gas as the working substance. With quantum mechanical systems as the working substance, however, a restriction on the two magnetic fields B_1, B_2, T_h and T_c is necessary, namely, these parameters can not be chosen arbitrarily. Usually the cycle falls into the following two categories ((ii) and (iii)). (ii) $\Delta Q < 0$. In this case, the amount of heat Q_{bc} flowing from the working substance to the regenerator in one regenerative process is larger than that (i.e., Q_{da}) flowing from the regenerator to the working substance in the other regenerative process. This requires that the redundant heat in the regenerator must be released timely to the cold heat bath, otherwise the regenerator may not operate normally. The release of heat for the regenerator increases the amount of heat release to the heat bath T_c. (iii) $\Delta Q > 0$. In contrary to the case (ii), the amount of heat flowing from the working substance to the regenerator Q_{bc} is smaller than that Q_{da} flowing from the regenerator to the working substance. The inadequate heat in the regenerator must be compensated from the heat bath T_h. This results in the increasing of heat absorbed from heat bath at high temperature. Hence the heat absorbed from heat bath T_h per cycle can be obtained as:

$$Q_h = Q_1 + \delta\Delta Q, \tag{9}$$

where $\delta = 0$ for $\Delta Q < 0$ and $\delta = 1$ for $\Delta Q > 0$.

We first discuss the positive work condition for quantum Stirling cycle. It has been shown that [3,4] the positive work condition of quantum Otto heat engine with a single spin as the working substance is $T_h > \frac{B_2}{B_1}T_c$, which is more restrictive than that in the classical Otto heat engine. Here we can prove that the positive work condition of quantum Stirling heat engine with a spin-$\frac{1}{2}$ particle as the working substance is $T_h > T_c$ and $B_1 < B_2$, which is consistent with the classical one. The proof for this condition is given in Appendix A.

Next we consider the heat engine efficiency. It can be obtained as:

$$\eta = \frac{W}{Q_h}. \tag{10}$$

The analytical expression of the engine efficiency is complicated, to show it clearly, we plot the numerical result in Figure 2. For classical Stirling heat engine, the efficiency is $\eta = 1 - \frac{T_c}{T_h}$, which is the same as that of Carnot efficiency

η_C and is independent of the generalized coordinate of the system (volume for $P-V-T$ system). With quantum system as the working substance, this observation does not hold. We observe from the figure that, the efficiency depends on both B_1 and B_2. In the figure, we use the ratio of B_2 to B_1 (namely, B_2/B_1) as the axis. This ratio can be seen as an isothermal compression in quantum systems. In most regime, there is an optimal isothermal compression for a fixed B_1, the efficiency reaches the peak, which corresponds to the condition for perfect regeneration. In other words, when $\Delta Q = 0$, the efficiency arrives in its maximum and this maximum is usually higher than the Carnot efficiency η_C. We should note that this result does not violate the second law of thermodynamics, as such a high efficiency profit is obtained from the effects of the regenerator. In order to work normally, the regenerator needs to consume additional energy. If we remove the regenerator, the heat engine cycle can still operate normally. However, the efficiency would decrease and it is lower than the Carnot efficiency. We should emphasize that the efficiency for a classical Stirling cycle with optimal regenerator equals to the Carnot efficiency. Our results show that the use of the regenerator can improve the efficiency of a quantum thermodynamical cycle higher than the Carnot efficiency.

In the following we focus on the inverse cycle of the Stirling heat engine. We should note that our heat engine is reversible. The proof is given in Appendix B. In the frame of classical thermodynamics, it is proved that the inverse cycle for a heat engine is a refrigeration cycle. However, this is not the case for quantum heat engine. Similar to the heat engine cycle, we start from point A in Figure 1. $A \to D$ is an isochoric process. The heat exchange $Q_{ad} = -Q_{da}$. Other processes can be described similarly with $Q_{cb} = -Q_{bc}$, $Q_1' = -Q_1$, $Q_2' = -Q_2$. The schematic diagram is the same as in Figure 1 but all arrows point to opposite direction as in Figure 1. The total work can be obtained as $W' = -W$. According to the positive work condition discussed above, we have $W' < 0$ when $T_c < T_h$ and $B_1 < B_2$. This indicates that the work is done on the working substance. The net amount of heat transfer between the system and the regenerator in the two isochoric processes is $\Delta Q' = Q_{ad} + Q_{cb}$. When $\Delta Q' < 0$, the amount of heat Q_{ad} flowing into the regenerator in one regenerative process is larger than that (Q_{cb}) flowing out from the regenerator in the other regenerative process. The redundant heat in the regenerator must be released to the heat bath T_c. This results in the reduction of the net amount of heat absorption from the hot bath T_c. Hence the total heat absorption from cold bath per cycle is:

$$Q_c = Q_2' - \delta|\Delta Q'| \tag{11}$$

where $\delta = 1$ for $\Delta Q' < 0$ and $\delta = 0$ for $\Delta Q' > 0$. We can see that if $\Delta Q' < 0$ and $Q_2' < |\Delta Q'|$, we have $Q_c < 0$. This means that the heat is released to the cold bath. In this case, the inverse cycle is not a refrigerator. In the case of $Q_c > 0$, the coefficient of performance for the refrigerator is

$$\varepsilon = \frac{Q_c}{|W'|}. \tag{12}$$

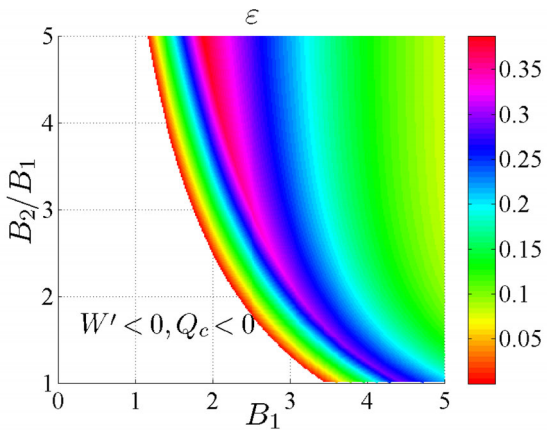

Fig. 3. Coefficient of performance ε as functions of B_1 (in units of kT_c) and B_2/B_1. Other parameters are $kT_h = 3kT_c$.

Figure 3 shows the dependence of the coefficient of performance on B_1 and the isothermal compression B_2/B_1. In this figure, the blank region corresponds to $W' < 0$ and $Q_c < 0$. As we discussed, the cycle is not a refrigerator in this regime. The condition that the cycle in this regime is $Q_2' + \Delta Q' < 0$. In addition, the coefficient of performance depends on B_1 and B_2, which is also different from the classical result $\varepsilon = \frac{T_c}{T_h - T_c}$. Compared with Figure 2, we find the maximum stands for the condition for perfect regeneration, i.e., when $\Delta Q' = 0$, the coefficient of performance reaches its maximum and this maximum is usually lower than the classical counterpart.

To shed more light on the condition with that the inverse cycle of a Stirling heat engine is not a refrigeration cycle, we consider the case of $\beta_h = 0$ and $B_1 = 0$, which means that the temperature of the hot bath is very high, and the energy of the lower level of the spin is zero. With these conditions, we have:

$$Q_c = -kT_c \ln\cosh(\beta_c B_2/2) \leq 0, \tag{13}$$

which is equal to zero when $\beta_c B_2 = 0$. This is illustrated by the blank region very close to the horizontal axis in Figure 3.

This observation tells us that with a spin-$\frac{1}{2}$ as the working substance, the inverse cycle of a Stirling heat engine may not be a refrigerator. This is totally different from that in classical physics. This feature is not rare, in fact with an ion trapped in a shallow potential, i.e., a harmonic oscillator with finite energy-levels, the inverse cycle of a Stirling heat engine is not a refrigerator, too. See, Figures 4a–4c. However, if the trapped potential is infinitely high, namely, the harmonic oscillator has infinite energy levels, the inverse cycle is a refrigerator, numerically described as Figure 4d, which comes from the following

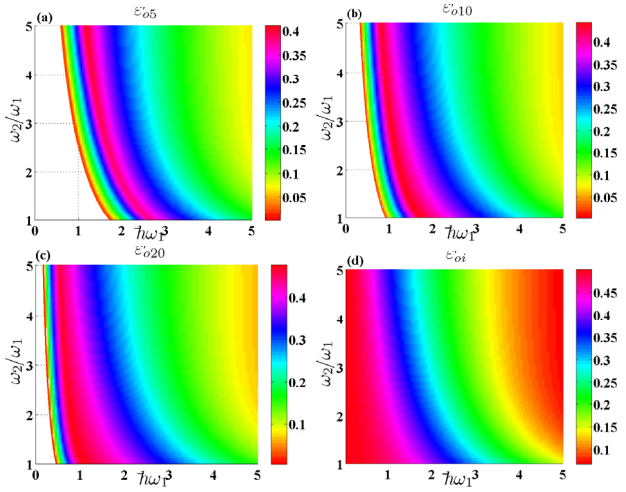

Fig. 4. Coefficient of performance ε as functions of $\hbar\omega_1$ (in units of kT_c) and ω_2/ω_1. Other parameters are $kT_h = 3kT_c$. The figures illustrate the heat engine composed of the working substance having five (a), ten (b), twenty (c), and infinite (d) energy levels.

calculations:

$$\begin{aligned} Q_2' &= -T_c k \ln \frac{\operatorname{csch}\left(\frac{1}{2}\hbar\omega_2\beta_c\right)}{\operatorname{csch}\left(\frac{1}{2}\hbar\omega_1\beta_c\right)} - \frac{1}{2}\hbar\omega_2 \coth\left(\frac{1}{2}\hbar\omega_2\beta_c\right) \\ &\quad + \frac{1}{2}\hbar\omega_1 \coth\left(\frac{1}{2}\hbar\omega_1\beta_c\right), \\ \Delta Q' &= -\frac{1}{2}\hbar\omega_1 \coth\left(\frac{1}{2}\hbar\omega_1\beta_c\right) + \frac{1}{2}\hbar\omega_1 \coth\left(\frac{1}{2}\hbar\omega_1\beta_h\right) \\ &\quad - \frac{1}{2}\hbar\omega_2 \coth\left(\frac{1}{2}\hbar\omega_2\beta_h\right) + \frac{1}{2}\hbar\omega_2 \coth\left(\frac{1}{2}\hbar\omega_2\beta_c\right), \\ W' &= T_h k \ln \frac{\operatorname{csch}\left(\frac{1}{2}\hbar\omega_2\beta_h\right)}{\operatorname{csch}\left(\frac{1}{2}\hbar\omega_1\beta_h\right)} + T_c k \ln \frac{\operatorname{csch}\left(\frac{1}{2}\hbar\omega_1\beta_c\right)}{\operatorname{csch}\left(\frac{1}{2}\hbar\omega_2\beta_c\right)}. \end{aligned} \tag{14}$$

3 Coupled spins

Now we consider the coupled spins as the working substance. The Hamiltonian of the working substance under consideration can be written as:

$$H = \frac{B}{2}\left(\sigma_1^z + \sigma_2^z\right) + J\left(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+\right) \tag{15}$$

where J is the coupling constant between the spins. $J > 0$ and $J < 0$ correspond to the antiferromagnetic and the ferromagnetic case, respectively. Here we only consider the antiferromagnetic case, i.e., $J > 0$. The four eigenvalues and corresponding eigenstates can be easily obtained as:

$$\begin{aligned} &E_1 = -B, : \quad |\psi_1\rangle = |00\rangle, \\ &E_2 = -J, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \\ &E_3 = J, \quad |\psi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ &E_4 = B, \quad |\psi_4\rangle = |11\rangle. \end{aligned} \tag{16}$$

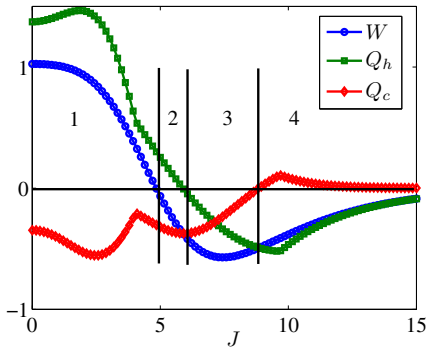

Fig. 5. Heat absorption Q_h from bath at high temperature T_h, heat release Q_c to bath at low temperature T_c and net work output as a function of coupling constant J (in units of kT_c). Other parameters for the cycle are $kT_h = 3kT_c$, $B_1 = 3kT_c$, $B_2/B_1 = 2$.

For the coupled spins as the working substance, we take the coupling constant as a parameter and keep it as a constant during the cycle. We consider the heat engine cycle only. The cycle process is also shown in Figure 1 and the cycle direction is $ABCDA$. The heat exchange in each stage is similar to equation (6) except for a replacement for the internal energy and entropy as:

$$U = -\frac{B\sinh B\beta + J\sinh J\beta}{\cosh B\beta + \cosh J\beta}, \tag{17}$$

$$S = k\left\{\ln[2(\cosh B\beta + \cosh J\beta)] + \frac{U}{kT}\right\}. \tag{18}$$

As a result, the net work output per cycle can be written as:

$$\begin{aligned} W = kT_h \ln\frac{\cosh B_1\beta_h + \cosh J\beta_h}{\cosh B_2\beta_h + \cosh J\beta_h} \\ + kT_c \ln\frac{\cosh B_2\beta_c + \cosh J\beta_c}{\cosh B_1\beta_c + \cosh J\beta_c}. \end{aligned} \tag{19}$$

For this net work output, $W > 0$ and $W < 0$ denote the work is done by and on the working substance. The net amount of heat transfer between the system and regenerator is $\Delta Q = Q_{bc} + Q_{da}$. With a single spin as the working substance, the perfect regeneration condition $\Delta Q = 0$ is difficult to met for arbitrary B_1, B_2, T_c, and T_h. However, for coupled spins, this condition can be satisfied for arbitrary B_1, B_2, T_c and T_h by adjusting the coupling constant J. In general, for fixed B_1, B_2 and T_c, T_h, there are two coupling constants J that can enforce $\Delta Q = 0$. We will see in the following that these two points correspond to two different cases.

Next we consider the heat absorption Q_h from the bath at high temperature T_h, heat release Q_c to the bath at low temperature T_c and net work output W. As discussed above, when $\Delta Q \neq 0$, Q_h and Q_c are determined by Q_1, Q_2, and ΔQ. The numerical results are shown in Figure 5. Here we calculate these thermodynamical quantities as a function of coupling constant J (in units of kT_c). There

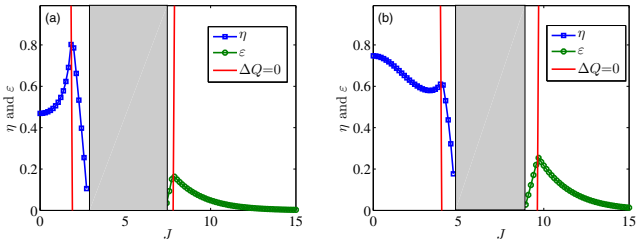

Fig. 6. Efficiency or coefficient of performance as a function of coupling constant J (in units of kT_c). Other parameters in the cycle are: $kT_h = 3kT_c$, $B_2/B_1 = 2$. Left: $B_1 = kT_c$; right: $B_1 = 3kT_c$. The grey regimes in the figures correspond to the cases of neither heat engine nor refrigerator.

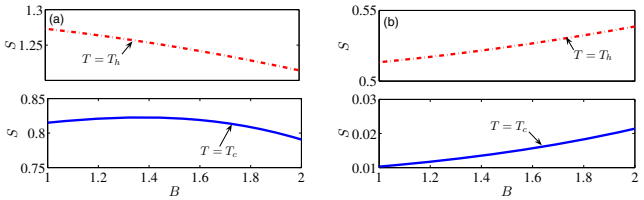

Fig. 7. The isothermal line in the magnetic field-entropy (B-S) plane. The parameters in the figure are same to Figure 6a. In Figure 6a, the two coupled constants correspond to $\Delta Q = 0$ are named as $\mathcal{J}_1$ and $\mathcal{J}_2$, where $\mathcal{J}_1$ corresponds to the maximal efficiency and $\mathcal{J}_2$ corresponds to the maximal coefficient of performance. In this figure, we have set: (a) $J = \mathcal{J}_1$; (b) $J = \mathcal{J}_2$.

are three vertical black lines in the figure, which divide the figure into four regimes, named $1 \sim 4$. In regime 1, we have $Q_h > 0$, $Q_c < 0$, and $W > 0$. This is the typical regime for the heat engine cycle. But it is not true for other regimes. In regime 2, we see $Q_h > 0$, $Q_c < 0$ and $W < 0$. This means the system absorbs heat from bath T_h, and releases heat to bath T_c with the work done on the working substance. This has no application in industry. In regime 3, we have $Q_h < 0$, $Q_c < 0$ and $W < 0$, i.e., the system releases heat to both bathes and a work is done on the working substance. This also has no use. In regime 4, we obtain $Q_h > 0$, $Q_c < 0$ and $W < 0$. This is the cycle of refrigeration. Hence we conclude that, when increasing the coupling constant J, we can turn a heat engine cycle into a refrigeration cycle.

The numerical results of efficiency and the performance coefficient are given in Figure 6. The two vertical red lines indicate $\Delta Q = 0$, one of them is in the regime of heat engine cycle, and the other is in the regime of refrigeration cycle. Moreover, we can see from the figure that when $\Delta Q = 0$, both the efficiency and the coefficient of performance reach a peak. The peak for the coefficient of performance is the maximum for refrigeration cycle, however, the peak for the efficiency is not always the maximal efficiency for the heat engine cycle. For a set of parameters B_1, B_2 and kT_c and kT_h, if the heat engine efficiency is lower than Carnot efficiency when $J = 0$, the peak is the maximal efficiency (Fig. 6a), otherwise it is not, see Figure 6b.

As shown, the coupling constant can turn a heat engine into a refrigeration cycle, the physics behind this conversion is as follows. We plot the isothermal lines in the magnetic field-entropy plane (B-S plane) for different coupling constants J. In this figure, other parameters are same as those in Figure 6a. There are two coupling constants corresponding to $\Delta Q = 0$ (vertical red line in Fig. 6a), which are marked by $\mathcal{J}_1$ and $\mathcal{J}_2$ with $\mathcal{J}_1 < \mathcal{J}_2$. In these two cases, the heat released in one regenerative process is absorbed entirely by the other regenerative process. Comparing Figures 7a and 7b, one can easily find that the monotonicity of the isothermal lines are different, i.e., when increasing the coupling constant J, the monotonicity of the isothermal lines change. As a result, the isothermal process with heat absorption becomes the isothermal process with heat release, and a heat engine cycle becomes a refrigeration cycle. We should note in Figure 7a that when $T = T_c$, the isothermal line is a non-monotonic curve. This means in an isothermal process with $T = T_c$, when we increase the magnetic field, the working substance absorbs heat first then releases heat shortly. The total result is that some heat is released to the cold bath.

4 Proposed realization in a Paul trap

The prediction can be observed in a single ion trapped in a shallow linear Paul trap. The shallow trap can support only finite energy levels, realized by shortening the length of the rf electrodes in the setup in reference [31]. In contrary to the proposal in reference [31], we replace the isentropic process which is nonthermal by the isothermal process, this can be done by keeping the system in contact with the bath. By changing the length of the rf electrodes, we can manipulate the number of levels in the trap and consequently control the regime in which the inverse cycle of the stirling heat engine is not a refrigerator.

5 Conclusions

In summary, we have considered the quantum Stirling cycle for a single spin or coupled spins as the working substance. The quantum Stirling cycle is the quantum mechanical generalization of the classical Stirling cycle which contains two quantum isothermal processes and two quantum isochoric processes. According to the quantum mechanical generalization of classical thermodynamical processes, in the quantum isothermal process, the working substance is kept contact with a heat bath and the external magnetic field is altered slowly so that the working substance is always in the thermal equilibrium state. We should notice that in classical isothermal process, the internal energy of the working substance (the ideal gas) does not depend on the generalized coordinate of the system (volume V). But this is not true for quantum working substance. This properties leads to new result for quantum Stirling cycle. In the quantum isochoric process, the external magnetic field is fixed. Our results show that due to

the properties of quantum mechanical system, the thermodynamical cycle exhibits many interesting properties which are quite different from the classical counterpart. For a single spin, both the heat engine cycle and refrigeration cycle are discussed. We have found that for heat engine cycle, the efficiency can exceed the Carnot efficiency. This improvement comes from the use of the regenerator. The regenerator is similar to the classical. As a result, the reason for such a high efficiency comes from the quantum mechanical properties of the working substance. Moreover we find that the inverse cycle of the heat engine is not always a refrigeration cycle. This holds true for a Stirling heat engine with an ion trapped in a shallow potential as the working substance. For the refrigeration cycle, the coefficient of performance is always lower than its classical counterpart. These results indicate that quantum Stirling cycle is more suitable for designing a heat engine but not advisable for refrigerator. With two coupled spins as the working substance, we find that the heat engine and the refrigeration cycle can be altered by increasing the coupling constant. This alternation can be explained as the transformation of the monotonicity for the isothermal line in magnetic field-entropy plane.

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Appendix A: The proof for the positive work condition

Here we give the detailed proof for the positive work condition. We first define a function as:

$$f(x) = \frac{[\cosh(x/2kT_h)]^{kT_h}}{[\cosh(x/2kT_c)]^{kT_c}}. \tag{A.1}$$

Taking derivative with respect to x, we have:

$$f'(x) = \frac{1}{2}\frac{[\cosh(x/2kT_h)]^{kT_h-1}}{[\cosh(x/2kT_c)]^{kT_c+1}}\sinh\frac{(kT_c-kT_h)x}{2kT_ckT_h}. \tag{A.2}$$

Here we assume that $T_c < T_h$, as a result, for $\forall\ x > 0$, we have $f'(x) < 0$. This means that $f(x)$ is a decreasing function. Consequently, if $B_1 < B_2$ we obtain $f(B_1) > f(B_2)$. This inequality leads to $W > 0$.

Appendix B: Proof of the the reversibility of the quantum Stirling cycle with single spin as working substance

According to the second law of thermodynamics, a reversible cycle should satisfy

$$\oint \frac{đQ}{T} = 0, \tag{B.1}$$

in a whole cycle. We consider one direction for the cycle, for example, heat engine cycle $ABCDA$ in Figure 1. The first step $A \to B$ is an isothermal process, in which $đQ = TdS$. As a result

$$\begin{aligned}\int_A^B \frac{đQ}{T} &= \int_A^B dS = S(B) - S(A)\\ &= k\ln\cosh\frac{\beta_hB_1}{2} - \frac{k\beta_hB_1}{2}\tanh\frac{\beta_hB_1}{2}\\ &\quad - k\ln\cosh\frac{\beta_hB_2}{2} + \frac{k\beta_hB_2}{2}\tanh\frac{\beta_hB_2}{2}.\end{aligned} \tag{B.2}$$

The second step $B \to C$ is an isochoric process, in which we have $đQ = \sum_n E_n dp_n$. We can obtain:

$$\int_B^C \frac{đQ}{T} = \sum_n E_n \int_B^C \frac{1}{T}\frac{dp_n}{dT}dT. \tag{B.3}$$

For a single spin, the two eigenvalues and the corresponding probabilities are $E_{1,2} = \pm\frac{1}{2}B_1$ and $p_{1,2} = \frac{1}{2}e^{\mp\frac{B_1}{kT}}\mathrm{sech}\frac{B_1}{2kT}$, respectively. Based on these results, we have:

$$\begin{aligned}\int_B^C \frac{đQ}{T} &= \frac{B_1^2}{4k}\int_{T_h}^{T_c}\frac{\mathrm{sech}^2\frac{B_1}{2kT}}{T^3}dT\\ &= k\ln\cosh\frac{\beta_cB_1}{2} - \frac{k\beta_cB_1}{2}\tanh\frac{\beta_cB_1}{2}\\ &\quad - k\ln\cosh\frac{\beta_hB_1}{2} + \frac{k\beta_hB_1}{2}\tanh\frac{\beta_hB_1}{2}.\end{aligned} \tag{B.4}$$

The following two steps can be calculated similarly as:

$$\begin{aligned}\int_C^D \frac{đQ}{T} &= k\ln\cosh\frac{\beta_cB_2}{2} - \frac{k\beta_cB_2}{2}\tanh\frac{\beta_cB_2}{2}\\ &\quad - k\ln\cosh\frac{\beta_cB_1}{2} + \frac{k\beta_cB_1}{2}\tanh\frac{\beta_cB_1}{2}\end{aligned} \tag{B.5}$$

$$\begin{aligned}\int_D^A \frac{đQ}{T} &= k\ln\cosh\frac{\beta_hB_2}{2} - \frac{k\beta_hB_2}{2}\tanh\frac{\beta_hB_2}{2}\\ &\quad - k\ln\cosh\frac{\beta_cB_2}{2} + k\frac{\beta_cB_2}{2}\tanh\frac{\beta_cB_2}{2}.\end{aligned} \tag{B.6}$$

Combining equations (B.2)–(B.6), we have $\oint \frac{đQ}{T} = 0$, i.e., the quantum Stirling cycle with single spin as working substance is reversible.

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